Technical Notes

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Improved Design Formulas for Buckling of Orthotropic Plates Under Combined Loading

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Nomenclature

D_{ij}	=	flexural stiffnesses of classical laminated-plate theory
K	=	buckling coefficient
N_b	=	stress resultant for maximum axial compression under bending
N_x	=	axial compression stress resultant
N_{xy}	=	shear stress resultant
R_i	=	critical load ratio or critical ratio of buckling coefficients
β	=	nondimensional orthotropic material parameter
δ, γ	=	nondimensional flexural anisotropy parameters

Subscripts

x, b, s	=	axial compression, bending, or shear, respectively
i	=	x, b, or s

Superscript

m	=	exponent in interaction formula

I. Introduction

THERE remains a need for more efficient use of materials in aircraft and spacecraft. For example, a small (e.g., 1%) weight saving for a wing skin material provides significant fuel savings over the lifetime of the aircraft. Similarly, the cost of placing structural

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weight in orbit is approximately 10, 000/lb. For structural members that are stiffness critical, the use of laminated-composite structures and the development of the corresponding refined buckling design formulas may possibly lead to significant weight savings. One structural member that is common in many aerospace structures is the flat plate. Often, for laminated-composite plates subjected to combined loads, buckling interaction data and simple design formulas do not exist and these members are designed by using the interaction formulas for isotropic plates [1–3]. This approach, when overly conservative, leads to a significant weight penalty. In contrast, when this approach is nonconservative, a significant impact to schedule may result. In this paper, a first step toward refined buckling interaction formulas for laminated-composite plates is presented. Specifically, refined buckling interaction formulas that have been obtained by using an accurate Rayleigh–Ritz analysis, and recently obtained bounds on the stiffness properties of composite materials, are presented for long orthotropic rectangular plates subjected to combined loads. It is important to note that results for long plates are applicable to plates of finite length for which the length-to-width ratio is approximately greater than three. For some cases, results for long plates provide a meaningful lower bound to the well-known buckling curves for finite-length plates that are plotted as a function of the length-to-width ratio. As such, the results presented herein are of practical importance. The combined-load cases include uniaxial compression (or tension) and shear (Fig. 1), pure bending and shear (Fig. 2), and uniform compression (or tension) and pure bending (Fig. 3). The results presented indicate the possibilities for significant weight savings. A description of the analysis and highlights of the key results are presented subsequently.

II. Background

Regarding combined uniaxial tension or compression and shear loads, the following interaction formula was derived by Chwalla [4] and also independently by Stowell and Schwartz [5]:

$$R_x + R_s^2 \leq 1 \qquad (1)$$

for long isotropic plates. In this equation, and others that follow herein, R represents the ratio of critical load for the combined-loading state to critical value of the corresponding load acting alone. The subscripts x and s refer to the axial and shear load, respectively. Stowell and Schwartz used an energy method to compute buckling coefficients for long isotropic plates and found that Eq. (1) gives excellent correlation, to within 1%, with their numerical results. In addition, this excellent correlation was obtained for simply supported and clamped boundary conditions. However, they found increasingly worse correlation for progressively increasing axial tension. These findings have been confirmed in the current work.

Timoshenko [6], Way [7], and Johnson and Buchert [8] produced buckling data for long isotropic plates subjected to combined pure in-plane bending and shear loads. The first two authors [6,7] presented their results with the interaction formula

$$R_b^2 + R_s^2 \leq 1 \qquad (2)$$

where the subscript b refers to pure in-plane bending. These authors claim Eq. (2) gives acceptable accuracy for both simply supported and clamped boundary conditions. This claim was assessed in the present study and it was found that Eq. (2) gives a conservative
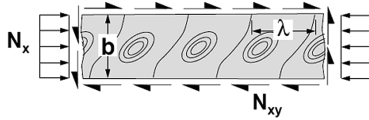
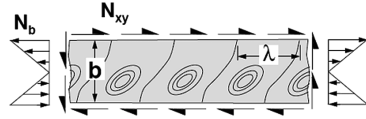**Fig. 1   Buckle pattern of long orthotropic plate under combined axial compression and shear loads.**

Fig. 2   Buckle pattern of long orthotropic plate under combined pure in-plane bending and shear loads.
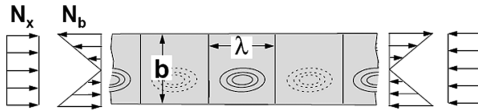

Fig. 3   Buckle pattern of long orthotropic plate under combined axial compression and pure in-plane bending loads.

estimate of buckling load to within an accuracy of 20% for isotropic plates. Because Johnson and Buchert [8] presented their results as buckling-coefficient charts, the accuracy is difficult to assess.

For the case of combined uniaxial tension or compression and pure in-plane bending loads, Noel [9] produced buckling-coefficient design charts, whereas Stüssi et al. [10] presented their results as

$$R_x + R_b^2 \leq 1 \tag{3}$$

The accuracy of Eq. (3) was also assessed in the present study for long isotropic plates with simply supported or clamped boundary conditions. For axial compression loads, the accuracy of Eq. (3) is within 2% of the corresponding analysis results obtained in the present study, but, for progressively larger axial tension loads, Eq. (3) becomes increasingly conservative and may underestimate buckling loads by 10%.

It is relevant to note that the three buckling interaction formulas, Eqs. (1–3), fit the general form

$$\sum_{i=1}^{n} R_i^{m_i} \leq 1 \tag{4}$$

where i represents a particular loading component, n is the number of different loading components, and the index on the exponent m indicates that it is a function of loading type. Although Eq. (4) has been applied in an empirical fashion to approximate the buckling response under combined loading, there is some underlying theoretical foundation for its use. Equation (4) has its roots in Dunkerley's formula [11] and indeed simplifies to this formula when $m_i = 1$ for all loading types. Dunkerley empirically derived his expression to obtain the overall natural frequency of a system of vibrating entities. Dunkerley's formula was later shown to provide a theoretical lower bound on overall natural frequency by Jeffcott [12]. The interaction formulas for natural frequencies were later applied to the buckling of flat plates under combined loading [4–10]. Furthermore, Murray [13] showed that Eq. (4), with $m_i = 1$, provides a lower bound to the buckling load of a long, flat, isotropic plate under combined loading. This conclusion was based upon prior work by Schaeffer [14] who had shown that the response surface, defined in terms of R_i ratios for an isotropic plate, is convex. As such, one might expect $m_i > 1$ to provide a better correlation with buckling response because a response surface with greater convexity is created. In contrast, values of $m_i < 1$ provide a concave response surface.

The empirical formulas presented in Eqs. (1–3) were derived by curve fitting numerical data for isotropic plates. The current work focuses on establishing values for the exponent m_i in Eq. (4) for long orthotropic plates, where $m_i > 1$, in a similar manner. In particular, a general form for the interaction formula given by

$$R_x^{m_x} + R_b^{m_b} + R_S^{m_s} \leq 1 \tag{5}$$

was investigated, in which any two permutations of the three loading components previously described were considered. To be useful in design, equations such as Eqs. (1–5) must be validated across the feasible design space. This type of validation does not appear to have been done for plates made of laminated-composite materials.

Recently, buckling formulas were presented for long orthotropic plates subjected to each single component of the combined-loading cases considered herein and with simply supported or clamped boundary conditions [15]. It was shown that the buckling coefficients are a monotonically increasing function of one nondimensional parameter β, given by

$$\beta = \frac{(D_{12} + 2D_{66})}{(D_{11}D_{22})^{1/2}} \tag{6}$$

where D_{ij} are flexural stiffnesses of classical laminated-plate theory. It was further shown that β has a well-defined range of values given by

$$-1 < \beta < 3 \tag{7}$$

with the further limitation that $\beta > 0$ for presently known material systems. Generally, β takes its smallest values for cross-ply laminates and its maximum value for angle-ply laminates with 45-deg ply angles. Also, $\beta = 1$ for isotropic materials and monotonically approaches $\beta = 1$ for quasi-isotropic layups as the number of plies increases [16]. Therefore, it is possible to produce validated interaction formulas, of the type indicated in Eq. (4), for a broad class of laminated plates by curve fitting results from numerical buckling analyses over the range $0 < \beta < 3$.

III.   Approach and Discussion of Results

The approach used in the present paper to obtain buckling interaction formulas for infinitely long orthotropic plates is based on the fact that the buckling coefficients are functions of only β and the particular loading and boundary conditions. This fact is illustrated in Figs. 4–6 for plates subjected to axial compression or tension and shear, pure in-plane bending and shear, and axial compression or tension and pure in-plane bending loads, respectively. The results shown in these figures were obtained by using the nondimensional Rayleigh–Ritz method for infinitely long anisotropic plates developed and used extensively by Nemeth [16–22]. The ordinate and abscissa values in each of Figs. 4–6 correspond to ratios of the
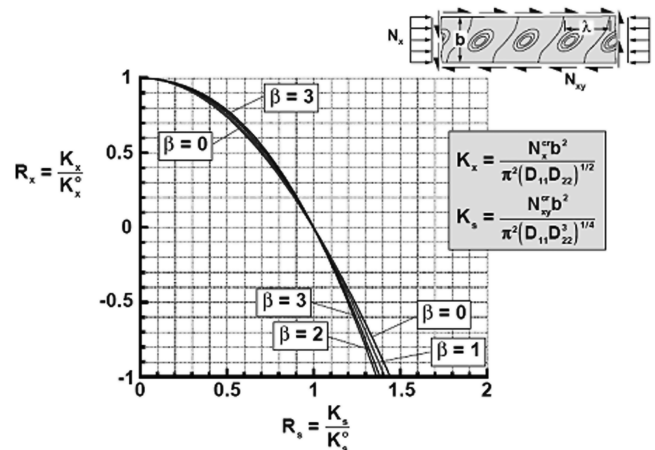

Fig. 4   Generic buckling interaction curves for long orthotropic plates subjected to uniaxial compression or tension and shear loads.
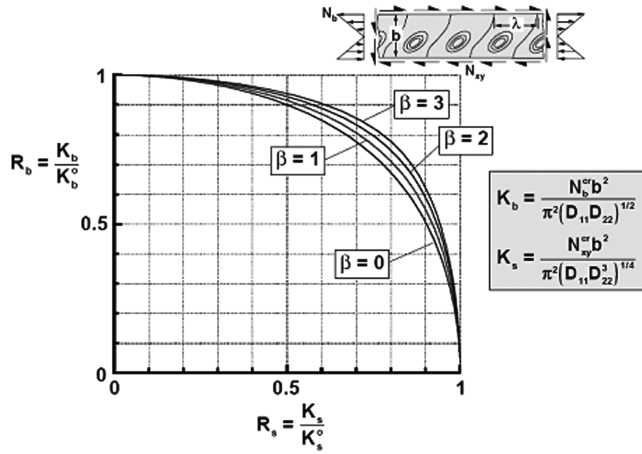
Fig. 5 Generic buckling interaction curves for long orthotropic plates subjected to pure in-plane bending and shear loads.

Fig. 6 Generic buckling interaction curves for long orthotropic plates subjected to uniaxial compression or tension and pure in-plane bending loads.

buckling coefficients. Buckling coefficients with the superscript "o" correspond to values associated with the implied single-component loading condition. The negative values of the buckling-coefficient ratios shown in Figs. 4 and 6 correspond to axial tension loads. Four curves are shown in each figure that correspond to values of $\beta = 0, 1, 2,$ and 3. In addition, it is noted that only two quadrants of the buckling response are shown in Figs. 4 and 6 because of symmetry of the buckling response of orthotropic plates under reversal of shear and pure in-plane bending loads. Similarly, only one quadrant of the
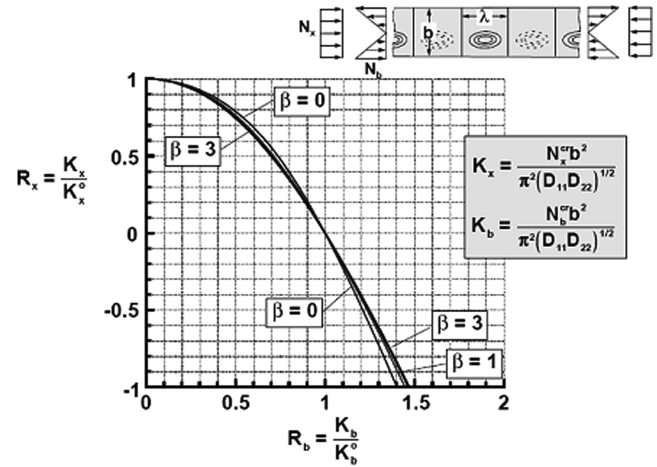
response is shown in Fig. 5. Points on and to the left of each curve correspond to stable equilibrium configurations.

The results in Figs. 4–6 indicate that the buckling interaction curves vary with β for each combined-loading case considered, as expected. The greatest variation is exhibited by the plates subjected to pure in-plane bending and shear loads, and the least variation occurs for combined compression or tension and pure in-plane
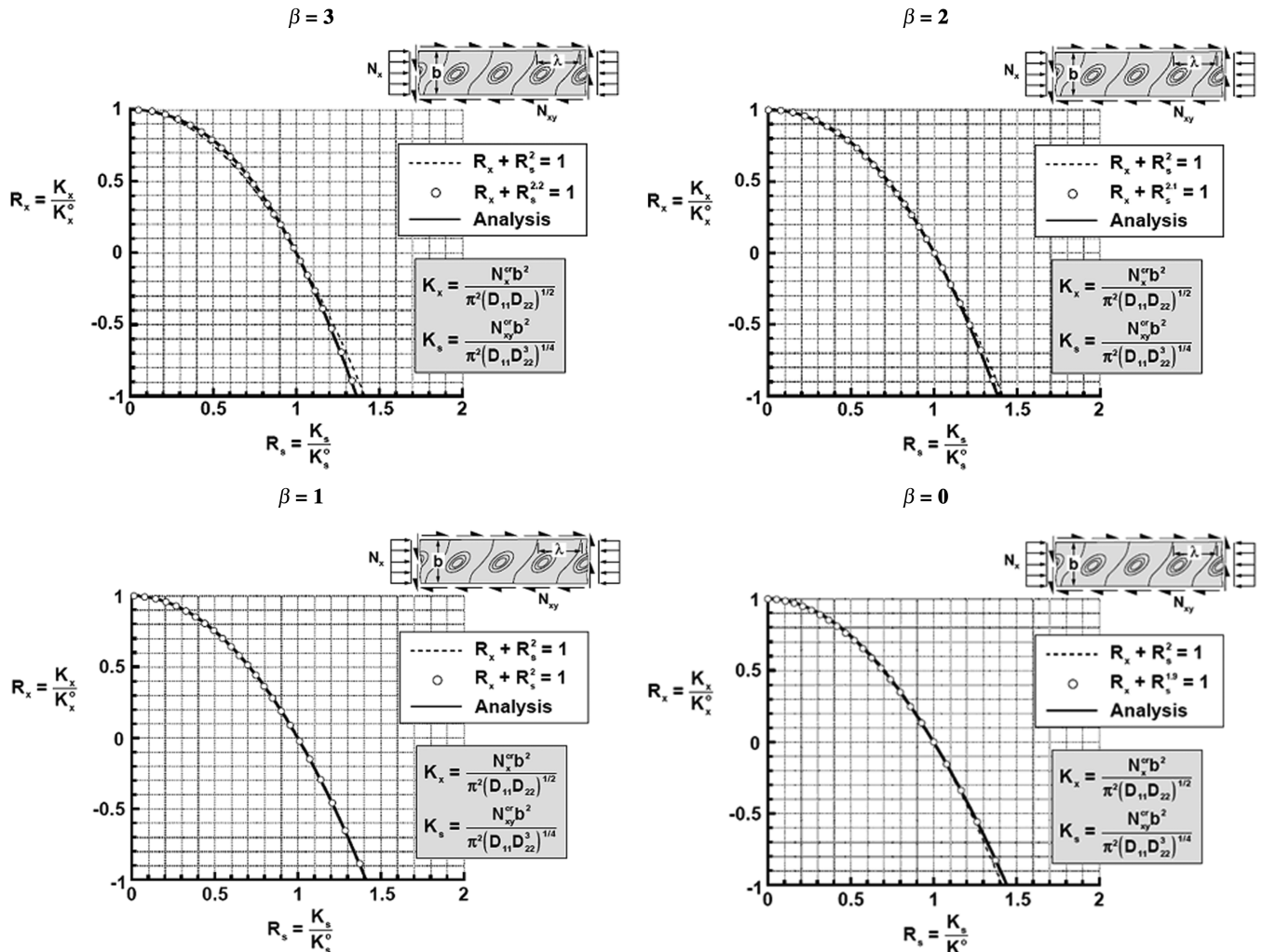

Fig. 7 Curve fit of generic buckling interaction curve for long, simply supported orthotropic plates with $\beta = 3, 2, 1,$ and 0 and subjected to uniaxial compression or tension and shear loads.
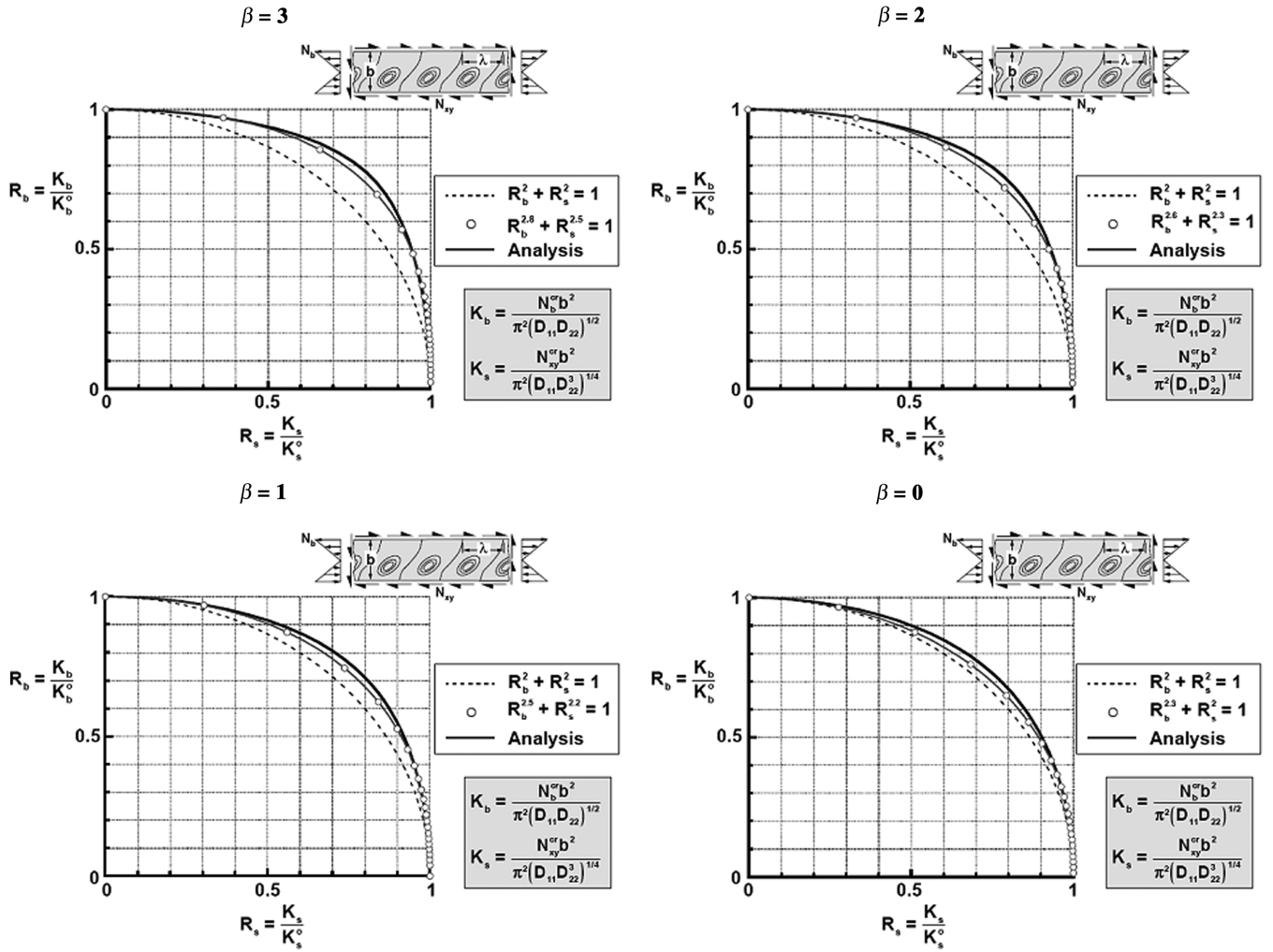
Fig. 8   Curve fit of generic buckling interaction curve for long, simply supported orthotropic plates with $\beta = 3, 2, 1,$ and 0 and subjected to pure in-plane bending loads and shear loads.

bending. Overall, the results in these figures led to use of the general buckling interaction formula given by Eq. (5), except with each exponent being a different linear function of the nondimensional parameter β. A linear function for each exponent was used to provide the simplest function that fits the analytical results to within a practical 5% difference. Furthermore, the exponents chosen were required to degenerate to the interaction formulas for long isotropic plates given by Eqs. (1–3). Finally, expressions for the exponents m_x, m_b, and m_s were determined by a least-squares fit to the numerical solutions obtained by using the nondimensional Rayleigh–Ritz method. Sample results obtained in the present study are shown in Figs. 7–9 for simply supported plates with $\beta = 3, 2, 1,$ and 0. The thick, solid line in these figures corresponds to the numerical solutions obtained by using the nondimensional Rayleigh–Ritz method and the thin, solid line with the circular symbols represents the corresponding curve fit. Dashed lines in the figures represent the corresponding isotropic-plate solution described previously herein and are included to illustrate potential pitfalls of their application to laminated-composite plates.

As a result of the studies used to obtain Figs. 7–9, the following interaction formulas were derived:

$$R_x + R_s^{1.9+0.1\beta} \leq 1 \qquad R_b^{\frac{13.8+\beta}{6}} + R_s^{\frac{12+\beta}{6}} \leq 1 \qquad R_x + R_b^2 \leq 1 \quad (8)$$

The formulas have been found to be valid for long orthotropic plates with either simply supported or clamped-edge conditions. It is demonstrated in Figs. 7–9 that the existing isotropic interaction

formulas may be up to 20% conservative for combined compression and shear loads, and that, for combined pure in-plane bending and shear loads, the degree of conservatism may be 25%. The modified interaction formulas, developed herein, show accuracies to within 5% of the Rayleigh–Ritz method, irrespective of loading combination, material system, or layup. However, the case of axial tension or compression and pure in-plane bending loads exhibits very little variation with orthotropy and, as a result, the formula for isotropic plates is sufficient for this loading case.

The formulas given by Eqs. (7) may also be applicable to symmetrically laminated-composite plates for which anisotropies associated with coupling between membrane and bending action (characterized by the $[B]$ matrix) and coupling between pure bending and twisting action (characterized by D_{16} and D_{26}) are negligible. It has been shown in [16–22] that the importance of neglecting flexural anisotropy on the buckling of long plates can be assessed by using the nondimensional parameters

$$\delta = \frac{D_{26}}{\sqrt[4]{D_{11}D_{22}^3}}, \qquad \gamma = \frac{D_{16}}{\sqrt[4]{D_{11}^3 D_{22}}}, \quad \text{and} \quad \beta = \frac{D_{12} + 2D_{66}}{\sqrt{D_{11}D_{22}}} \quad (9)$$

However, the results presented in [16–22] indicate that the importance of neglecting flexural anisotropy also depends substantially on loading conditions and boundary conditions. As a result, specific values of δ, γ defined by Eq. (9), for which Eqs. (8) remain valid, require an in-depth study that is beyond the scope of the present paper.
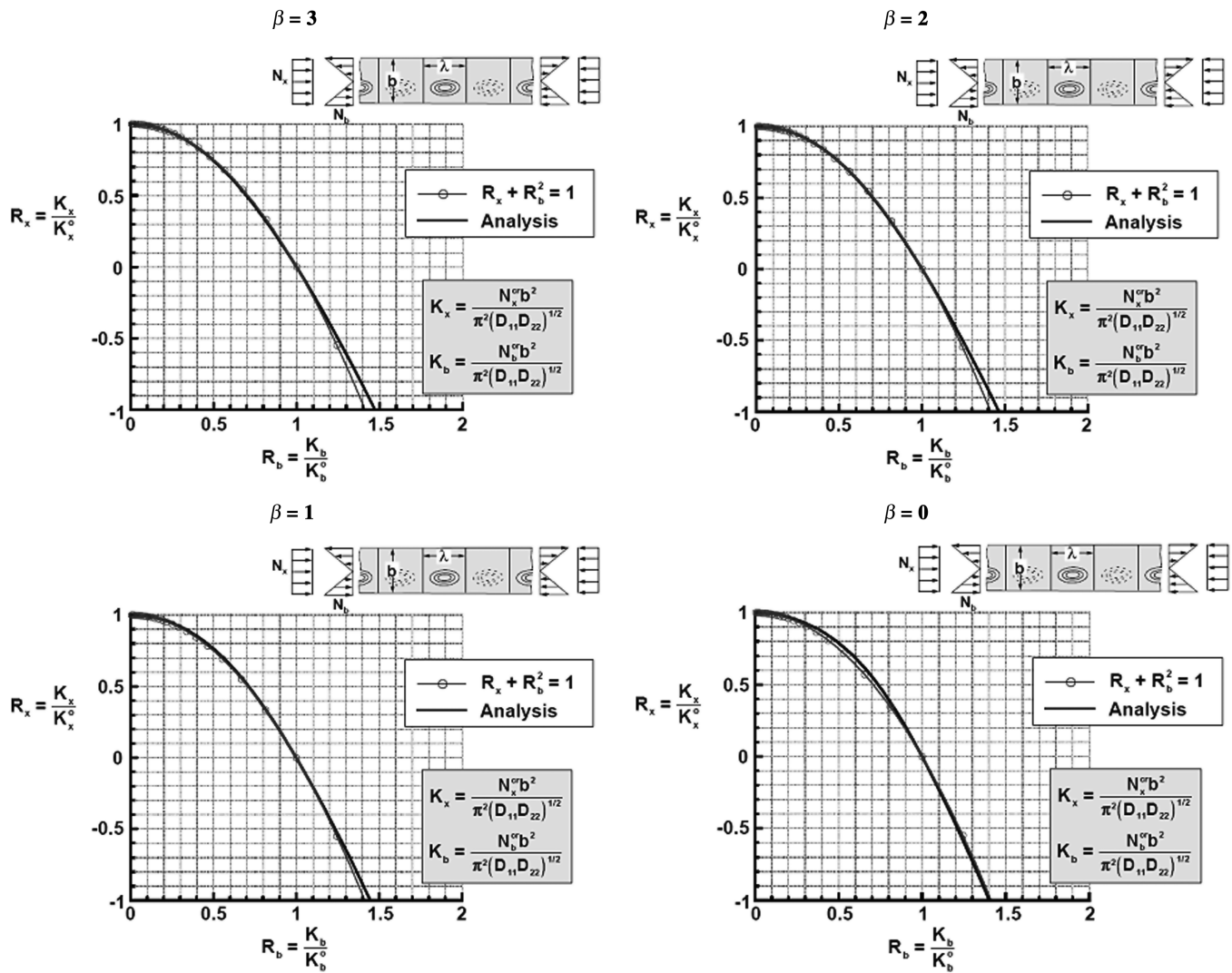
Fig. 9   Curve fit of generic buckling interaction curve for long, simply supported orthotropic plates with $\beta = 3, 2, 1$, and 0 and subjected to uniaxial compression (or tension) loads and pure in-plane bending loads.

IV.   Conclusions

Buckling interaction formulas have been derived for long orthotropic plates with simply supported or clamped edges and subjected to combined loading. Loading combinations include uniaxial tension or compression and shear, pure in-plane bending and shear, and uniaxial tension or compression and pure in-plane bending loads. These new formulas are an extension of the well-known formulas for isotropic plates that are sometimes applied erroneously to orthotropic and more general symmetrically laminated-composite plates. The results demonstrate the accuracy of the new formulas compared with highly accurate results obtained by using a Rayleigh–Ritz analysis method. As such, the new formulas will facilitate high-quality rapid design and optimization studies, and should be of great interest to designers and engineers.

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